1er Parcial

- Clásica → Puntos con sus imágenes
- No Clásicas → Puntos o derivadas en puntos y sus imágenes
  - Hermite: Puntos, Imagen de Puntos, Derivada en Puntos
    $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{2n+1} x^{2n+1}$
    $p'(x) = a_1 + 2a_2 x + \dots + (2n+1) a_{2n+1} x^{2n}$
  - Taylor → $x_0 \quad f(x_0) \quad f'(x_0) \quad f''(x_0) \dots$

1 CLÁSICA O DE LAGRANGE

Soporte ⇒ $S = \{x_0, x_1, \dots, x_n\}$ → Imágenes → $n+1$ datos → podemos resolver $n+1$ condiciones → max grado n

$\mathcal{P}_n$ ≡ Espacio vectorial de los polinomios de grado $\leq n$ → $\mathrm{Dim}\, \mathcal{P}_n = n+1$

Si nº Condiciones = Dim espacio → Sol. es única.

- Determinante de Van der Monde

$$\Delta = \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix}$$

- $\Delta \neq 0 \rightarrow \exists$ Sol y es única
- $\Delta = 0$ →
  - $Rg\,\Delta \neq Rg\,\bar{\Delta}$ → No Sol. → sistema incompatible
  - $Rg\,\Delta = Rg\,\bar{\Delta}$ → ∞ Sol. → Compatible indeterminado

2 POLINOMIO DE LAGRANGE

Sea $f(x)$ definida por $\{x_0, x_1, \dots, x_n\}$, $\{z_0, z_1, \dots, z_n\}$ Tomando $x_i \neq x_j \quad i \neq j$

- Polinomio i-ésimo → Polinomio l_i de grado no superior a n, tal que en x_i del soporte tome valor 1 y en todos los demás 0. → $\boxed{l_i \in \mathcal{P}_n \,/\, l_i(x_i) = 1,\ l_i(x_j) = 0 \ ; x_i \neq x_j}$ $P_i(x_j) = 0 \Rightarrow$ Raíz

$$\boxed{l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} \quad i = 0, 1, \dots, n}$$ → Polinomio de Lagrange → Base de Lagrange → $B\{l_0, \dots, l_n\}$

$$\boxed{P_n(x) = f(x_0) l_0(x) + \dots + f(x_n) l_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)}$$ → Fórmula de Lagrange para el caso de interpolación clásica

3 POLINOMIAL DE TAYLOR

$x_0 \in \mathbb{R}$, $f \in C^n(\{x_0\})$ Entonces $p \in \mathcal{P}_n \,/\, p^{(i)}(x_0) = f^{(i)}(x_0) \quad i = 0, \dots, n$

4 BASE DE LAGRANGE

$B = \{v_0, v_1, \dots, v_n\}$ → Cumpliendo ⇒ $L_i(v_i) = \begin{cases} 1 & \text{si } i = j \\ 0 & \text{si } i \neq j \end{cases}$

$V = \sum_{i=0}^{n} \alpha_i v_i$ → $L_0(V) = z_0,\ L_1(V) = z_1,\ L_2(V) = z_2$
$\alpha_0 = z_0,\ \alpha_1 = z_1,\ \alpha_2 = z_2$
$V = z_0 v_0 + z_1 v_1 + z_2 v_2$

5 Error de interpolación

$$|E(x)| = |P_n(x) - f(x)|\ x \in [a,b] \rightarrow \frac{|f^{(n+1)}(\xi_x)|}{(n+1)!} |(x - x_0)(x - x_1) \cdots (x - x_n)|$$

6 Base de NEWTON, POLINOMIO DE NEWTON Y DIFERENCIAS DIVIDIDAS

- Base de Newton caso clásico → $\{1,\ x - x_0,\ (x - x_0)(x - x_1), \dots, [(x - x_0)(x - x_1) \dots (x - x_{n-1})]\}$

$\Delta = |\Delta|$ → Diagonal principal $\neq 0$ $(x_i \neq x_j\ \forall j, i) \Rightarrow \exists$ Sol y es única

- Polinomio de Newton Caso clásico → $P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_n][(x - x_0)(x - x_1) \dots (x - x_{n-1})]$

- Diferencias Divididas → $f[x_0] = f(x_0)$ → $f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$ → $[x_0, x_1, x_2] = \dfrac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2}$

$* \ f[\underbrace{x_0, x_0, x_0}_{n}] = \dfrac{f^{(n-1)}(x_0)}{(n-1)!}$ ⇒ La utilización de Newton para casos no clásicos (Taylor, Hermite)

Hermite → $P_{2n+1}(x) = f[x_0] + f[x_0, x_0](x - x_0) + f[x_0, x_0, x_1](x - x_0)^2 + f[x_0, x_0, x_1, x_1](x - x_0)^2 (x - x_1) + \dots + f[x_0, x_0, x_1, x_1, \dots, x_n, x_n][(x - x_0)^2 (x - x_1)^2 \dots (x - x_{n-1})^2 (x - x_n)]$

$3n-1$
Grado 2 ⇒ 1 Lib
Grado 3 ⇒ 2 Lib
Splines
$4n-2$

Grado m → $\varphi_{i[x_{i-1}, x_i]}$ es un polinomio de grado m. → $\varphi \in C^{m-1}_{[a,b]}$ (Tiene $m-1$ derivadas continuas)

Exigir → Continuidad de $\varphi(x)$ ($n-1$ condiciones) → Continuidad de $\varphi'(x)$ ($n-1$ condiciones) → cond. interpolación ($n+1$ condiciones)

- Cúbicos Naturales ⇒ $S''(x_0) = S''(x_n) = 0$ (z_0, z_n) → Como tenemos z_0 y z_n → $\delta_i = |x_i - x_{i+1}|$ → y_i = Imagen en i

$$\frac{\delta_i}{6} z_{i-1} + \frac{\delta_i - \delta_{i+1}}{3} z_i + \frac{\delta_{i+1}}{6} z_{i+1} = \frac{y_{i-1}}{\delta_i} - \left(\frac{1}{\delta_i} + \frac{1}{\delta_{i+1}}\right) y_i + \frac{y_{i+1}}{\delta_{i+1}} \Rightarrow \text{obtener } z_i$$

$$S_i(x) = \frac{z_{i-1}}{6\delta_i}(x_i - x)^3 + \frac{z_i}{6\delta_i}(x - x_{i-1})^3 + \left(\frac{y_i}{\delta_i} - \frac{z_i \delta_i}{6}\right)(x - x_{i-1}) + \left(\frac{y_{i-1}}{\delta_i} - \frac{z_{i-1}\,\delta_i}{6}\right)(x_i - x)$$

$$\boxed{\begin{array}{c}\text{La recta} \\ \dfrac{x - x_0}{x_1 - x_0} = \dfrac{y - y_0}{y_1 - y_0}\end{array}}$$

MEJOR APROXIMACIÓN

Dist → $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$; Dist Euclídea → $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

i) $d(x,y) \geq 0 \quad \forall x, y \in E$
ii) $d(x,y) = d(y,x)$
iii) $d(x,y) \leq d(x,z) + d(y,z) \quad \forall x, y, z \in E$
Desigualdad triangular
Dist (E, d) → Espacio Métrico

i) $\|x\| > 0$; $\|x\| = x \iff x = 0$
ii) $\|\lambda x\| = |\lambda|\,\|x\| \quad \forall \lambda \in \mathbb{R}, x \in E$
iii) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in E$
$(E, \|\ \|)$ → Espacio Normado

$\langle x, y\rangle = \langle y, x\rangle \quad \forall x, y \in E$
i) $\langle \alpha x_1 + \beta x_2, y\rangle = \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$
ii) $\langle x, x\rangle \geq 0 \iff x = 0$

- Toda norma induce a una distancia: $d(x,y) = \|x - y\|$ → Todo espacio métrico es un espacio normado

- $\|x\| = \sqrt{\langle x, x\rangle}$
- $d(x,y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}$

• Producto escalar $\to$ $\langle f,g\rangle = \int_a^b f(x)\,g(x)\,dx$ • Norma $\|f\| = \sqrt{\langle f,f\rangle} = \sqrt{\int_a^b f(x)\,f(x)\,dx} \to \|g\| = \left(\int_a^b g^2(x)\,dx\right)^{1/2}$

• Distancia $\to$ $d\langle f,g\rangle = \|f-g\| = \left(\int_a^b (f(x)-g(x))^2\,dx\right)^{1/2}$ $u \to$ Mejor aproximación $\Rightarrow d(u,f) \le d(v,f) \ \forall v \in H$

• $u \in H$ es mejor aprox de f en $H \iff \langle f-u, v\rangle = 0 \ \forall v \in H$

– PROCESO DE GRAM-SCHMIDT $\to$ $\boxed{\tilde{v}_k = v_k - \frac{\langle v_k, \tilde{v}_1\rangle}{\langle \tilde{v}_1, \tilde{v}_1\rangle}\tilde{v}_1 - \dots - \frac{\langle v_k, \tilde{v}_{k-1}\rangle}{\langle \tilde{v}_{k-1}, \tilde{v}_{k-1}\rangle}\tilde{v}_{k-1}}$ $B \equiv \{v_1, v_2, \dots, v_n\}$ $\tilde{B} \equiv \{\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n\}$

• BASE ORTOGONAL

$f, g \in E$ ortogonales $\to \langle f,g\rangle = 0 \Rightarrow u(x) = \alpha_1\tilde{v}_1 + \alpha_2\tilde{v}_2 + \dots + \alpha_n\tilde{v}_n \to \alpha_1\langle\tilde{v}_1,\tilde{v}_1\rangle = \langle f,\tilde{v}_1\rangle; \dots; \alpha_n\langle\tilde{v}_n,\tilde{v}_n\rangle = \langle f,\tilde{v}_n\rangle$

$\boxed{\alpha_i = \frac{\langle f,\tilde{v}_i\rangle}{\langle\tilde{v}_i,\tilde{v}_i\rangle} = \frac{\langle f,\tilde{v}_i\rangle}{\|\tilde{v}_i\|^2} \quad i = 1\dots n}$ $\boxed{u(x) = \sum_{i=1}^n \alpha_i\tilde{v}_i = \sum_{i=1}^n \frac{\langle f,\tilde{v}_i\rangle}{\|\tilde{v}_i\|^2}\tilde{v}_i \quad i = 1\dots n}$ $\Rightarrow$ Mejor Aproximación con Base Ortogonal

• BASE ORTONORMAL

① $\to$ Ortogonal: $\langle v_i, v_j\rangle = 0 \quad i \ne j$; ② $\to$ Norma $= 1$: $v_i \ \|v_i\| = 1$; $\|v_i\|^2 = \langle v_i, v_i\rangle \Rightarrow \langle v_i, v_j\rangle = \delta_{ij} \quad i,j = 1\dots n$

$\boxed{u(x) = \sum_{i=1}^n \langle f, v_i\rangle v_i}$ $\Rightarrow$ Mejor aproximación con Base Ortonormal

$\to$ Suma de Fourier para f. $\langle f, v_i\rangle$ coeficiente de Fourier.

$\boxed{\begin{array}{l} \text{Si } \|v_i\| = 1 \\ \langle v_i, v_i\rangle = 1 \\ \text{Si } v_1 \perp v_2 \\ \langle v_1, v_2\rangle = 0 \end{array}}$

• CALCULO DE $u(x)$

1ª
i) Sacar una Base
ii) $\langle f-u, v_1\rangle = \langle f-u, v_2\rangle = \dots = \langle f-u, v_n\rangle = 0$
iii) $\langle u, v_1\rangle = \langle f, v_1\rangle; \dots; \langle u, v_n\rangle = \langle f, v_n\rangle$
iv) $u(x) = a_1v_1 + a_2v_2 + \dots + a_nv_n$
v) Sustituimos $u(x)$ en iii
vi) Calculamos cada a_i

2ª
i) Sacar $B\{v_1, v_2, \dots, v_n\}$ y exponer $\tilde{B}\{\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n\}$
ii) Doy valor a $\tilde{v}_1$ y saco $\langle v_1, v_2\rangle = 0$. Con $v_i = a_{1i}v_1 + a_{2i}v_2 + \dots + a_{ni}v_n$
iii) Con el paso ii obtengo todos los v_i (o GRAM-SCHMIDT)
iv) Con una base ortogonal genérica (iii) forzamos $\langle v_i, v_i\rangle = 1$; $\|v_i\| = 1$
v) $\tilde{v}_i \to \langle f-u, \tilde{v}_i\rangle = 0$. Con $u(x) = \alpha\tilde{v}_1 + \beta\tilde{v}_2 + \dots$
vi) calculamos $\alpha, \beta, \dots$

③ Mínimos cuadrados $\to$ Continuo
ⓐ $d(f,g) = \int_a^b (f(x)-g(x))^2\,w(x)\,dx$
ⓑ $\langle f,g\rangle = \int_a^b f(x)\,g(x)\,w(x)\,dx$
ⓒ $\|f\|^2 = \int_a^b f^2(x)\,w(x)\,dx$

$w(x) \to$ Función Peso
$w(x) \ge 0 \ \forall x$

$\to$ Discreto
ⓐ $d(f,g) = \|f-g\| = \left(\sum_{i=1}^n (f_i-g_i)^2 w_i\right)^{1/2}$
ⓑ $\langle f,g\rangle = \sum_{i=1}^n f_i g_i w_i$
ⓒ $\|f\|^2 = \sum_{i=1}^n f_i^2 w_i$

$w = (w_1, \dots, w_n)$
Dim $=$ nº vectores $\Rightarrow (v_1, v_2)$
Si $y = Ax + B \Rightarrow A\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} + B\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = \begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}$

$u(x)$ m.a $\iff \langle f-u, v_i\rangle = 0$

• ERROR DE MEJOR APROXIMACIÓN
$|e| = \|f-u\| = \sqrt{\|f\|^2 - \|u\|^2}$

$u(x) = cf(x)$

ECUACIONES NO LINEALES

① Bolzano $\to$ $f(a)\,f(b) < 0 \Rightarrow S \in [a,b]$ ② Unicidad $\to$ $\nexists f'(x)_{[a,b]} \ne 0 \Rightarrow s$ es Única en $[a,b]$

Cumpliendo ① y ②
• BISECCIÓN
$x_0 = \frac{a+b}{2}$
$|e_n| \le \frac{b-a}{2^{n+1}}$

• REGULA FALSI
$x_0 = \frac{af(b) - bf(a)}{f(b) - f(a)}$; $x_n = \frac{x_{n-1}f(b) - bf(x_{n-1})}{f(b) - f(x_{n-1})}$
$|e_n| = |x_n - x_{n-1}| \le \varepsilon \to$ Conocido

• SECANTE
$x_0 = a, x_1 = b$
$x_n = \frac{x_{n-2}f(x_{n-1}) - x_{n-1}f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}$

¡¡ SECANTE NO ASEGURA CONVERGENCIA !!

LINEALES

$d_1 = \sum_{i=1}^n x_i - y_i$ $\quad d_1(g(x), g(y)) = |g(x) - g(y)|$ $\quad \|x\|_1 = \sum_{i=1}^n |x_i|$

$d_2 = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$ $\quad d_2(g(x), g(y)) = \sqrt{|g(x) - g(y)|^2}$ $\quad \|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$

$d_\infty = \max|x_i - y_i|$ $\quad d_\infty(g(x), g(y)) = \max|g(x) - g(y)|$ $\quad \|x\|_\infty = \max|x_i|$

• Aplicación contractiva
$d(g(x), g(y)) = |g(x) - g(y)| \le K|x-y|$; $K < 1$
• Punto fijo de Aplicación Contractiva
$x = f(x) \iff x = \text{cte} \Rightarrow$ Pto. Fijo

Teorema del Punto Fijo $\to$ Si $g(x)$ es contractiva en un espacio métrico completo $\Rightarrow \exists s$ y es único. Además ese punto fijo es el lím de la sucesión $s = g(s)$ / $s = \lim_{n\to\infty}\{x_n\}$. Empezando en x_0 arbitrario

CONVERGENCIA GLOBAL $\to$ ① $g(x) \in C^1[a,b]$, $g([a,b]) \subset [a,b] \Rightarrow g(x) \in [a,b] \ \forall x \in [a,b] \Rightarrow \exists$ al menos un punto fijo $s \in [a,b]$

Si g creciente $\begin{cases} g(a) \ge a \\ g(b) \le b \end{cases}$ Si g decrece $\begin{cases} x_0 \in [a,b] \\ g(a) \le b \\ g(b) \ge a \end{cases}$

② Si $\exists$ una cte $L < 1$ / $d(g(x), g(y)) = |g(x) - g(y)| \le L|x-y| \Rightarrow s$ es único $\Rightarrow \lim_{n\to\infty}\{x_n\}$
$g(a) - g(b) = g'(\xi)(b-a) \Rightarrow |g(a) - g(b)| = |g'(x)|\,(b-a)$ $\to L < 1$

CONVERGENCIA LOCAL $\to$ ① Si $\exists s$ tal que $s = g(s)$ y $g(x) \in C^1[a,b]$ ② $|g'(s)| < 1$ } $x_0 \approx s$, $x_1 = g(x_0)$, $x_2 = g(x_1) \dots$ $g'(x_0) < 1 \dots |g'(x_n)| < 1 \Rightarrow$ Converge

ERROR EN LINEALES ITERATIVOS $\to$ $\boxed{|e_n| = |x_n - s| \le \frac{L^n|x_1 - x_0|}{1-L}}$; $L = \max|g'(x)|_{[a,b]}$

CONVERGENCIA LINEAL

① $\boxed{\begin{array}{l} s = g(s) \quad g'(s) \ne 0 \\ \lim_{n\to\infty}\frac{|e_{n+1}|}{|e_n|} = |g'(s)| \ne 0 \end{array}}$

Dem
x_0
$x_1 = g(x_0)$
$x_2 = g(x_1)$
$\vdots$
$x_{n+1} = g(x_n)$
$\to$ Desarrollo por Taylor

$g(x_n) = g(s) + g'(\xi)(x_n - s)$ con $\xi \in [x_n, s]$

$x_{n+1} - s = g'(\xi)(x_n - s) \Rightarrow \underbrace{|x_{n+1} - s|}_{e_{n+1}} = |g'(\xi)|\underbrace{|x_n - s|}_{e_n}$

$|e_{n+1}| = |g'(\xi)|\,|e_n| \Rightarrow \frac{|e_{n+1}|}{|e_n|} = |g'(\xi)| \to \xi \ne \text{cte} \to n$ altos ①

GRADO K

$s = g(s)$ $\quad g'(s) = g''(s) = \dots = g^{(k-1)}(s) = 0$

$g^{(k)}(s) \ne 0$ $\quad \lim_{n\to\infty}\frac{|e_{n+1}|}{|e_n|^k} = \frac{g^{(k)}(s)}{k!} \ne 0$

$\boxed{\text{NEWTON – RAPSHON}}$ → Se necesita convergencia local → $\boxed{x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}}$

- Condiciones de convergencia:

① $f(x) \in C^2 \Rightarrow$ Continua de orden 2 ② Bolzano $f(a)f(b)<0$ ③ $f'(x)_{[a,b]} \neq 0$ $\exists! s$ ④ $f''(x)_{[a,b]} \neq 0$

Comienzo a iterar con $x_0 \simeq s$ que cumpla

⑤ $f(x_0)\, f''(x_0) > 0 \Rightarrow \exists$ convergencia N-R ⑥ $\exists$ conv si $\max\left\{\left|\dfrac{f(a)}{f'(a)}, \dfrac{f(b)}{f'(b)}\right|\right\} \leq b-a$ ⑦ si $f(x) \in C^3$ al menos cuadrática

- Cota de Error

$|e_n| = |x_n - s| \leq \dfrac{M\,|x_n - x_{n-1}|^2}{2m}$; $M = \max |f''(x)|_{[a,b]}$; $m = \min |f'(x)|_{[a,b]}$

- Orden de Convergencia

$\lim\limits_{x\to\infty} \dfrac{|e_{n+1}|}{|e_n|^2} = \dfrac{f''(s)}{2f'(s)} = cte \neq 0$

$\boxed{\text{RAÍCES MÚLTIPLES}}$ → Si $s_1 = s_2 = \ldots = s_n$ → $x_n = x_{n-1} - p\,\dfrac{f(x)}{f'(x)}$

Si $s_1 \neq s_2$ → MÉTODOS SUSTITUTIVOS

① WITTACKER $x_n = x_{n-1} - \dfrac{f(x)}{m}$

m = orden de magnitud de $f'(x)_{[a,b]}$

② N-R MODIF $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_0)}$

③ Secante $x_n = \dfrac{x_{n-2} f(x_{n-1}) - x_{n-1} f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}$

$\boxed{\text{EJERCICIOS Y EJEMPLOS}}$ → Identidad de Parseval $\Rightarrow \langle f,g\rangle = \sum\limits_{i=1}^{N} \langle f,u_i\rangle\langle g,u_i\rangle$; con B ortogonal

$\left.\begin{array}{l} f(x) = a_1u_1(x) + a_2u_2(x) + \ldots + a_n u_n(x) \\ g(x) = b_1u_1(x) + b_2u_2(x) + \ldots + b_n u_n(x) \end{array}\right\}$, $x \in [0,\pi/2] \Rightarrow \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} a_ib_j\langle u_i,u_j\rangle =$ $\langle u_i,u_j\rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$= \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} a_ib_j\,\delta_{ij} = \sum\limits_{i=1}^{n} a_ib_i$ $\left|\begin{array}{l} \langle f,u_k\rangle = \sum\limits_{i=1}^{n} a_i\langle u_i,u_k\rangle = a_k \\ \langle g,u_k\rangle = \sum\limits_{i=1}^{n} b_i\langle u_i,u_k\rangle = b_k \end{array}\right\} \Rightarrow \sum\limits_{i=1}^{n} \langle f,u_k\rangle\langle g,u_k\rangle$

- me dan $u(x)$ m.a. ax y $f(x)$ $\operatorname{sen} x$ → $u(x) \Rightarrow a = \dfrac{\langle f,x\rangle}{\|x\|^2}$ $\int_0^{\pi/2} x \operatorname{sen} x\, dx = 1$ $\|x\|^2 = \langle x,x\rangle = \int_0^{\pi/2} x^2 dx$

Dominante en la diagonal $\rightarrow |a_{ii}| > \sum a_{ij}$ Def. ⊕ $\Rightarrow$ Lo de los determinantes $|A_k| > 0$ Simétrica ⊕ $x^T A x = 0 \Rightarrow x = 0$

Espectro Matriz $\rightarrow$ Conjunto de autovalores $\sigma(A) = \{\lambda_i(A)\}$ Radio espectral $\rightarrow \rho(A) = \max \{|\lambda_i|\}$ $\rho(A) \le \|A\|$ Semisimétrica ⊕ $\ge 0 \ \forall x \ne 0$

- Norma de vector
  $\|x\|_1 = \sum_{i=1}^{n} |x_i|$
  $\|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$
  $\|x\|_\infty = \max_{1 \le i \le n} \{|x_i|\}$

- Norma de una matriz
  $\|A\|_1 = \max \{|\text{suma columna}|\}$
  $\|A\|_2 = \sqrt{\rho(A^T A)}$
  $\|A\|_\infty = \max \{|\text{suma fila}|\}$ |cada elem|
  $E_i \longleftrightarrow E_j$ Filas
  $E_i \longleftrightarrow \lambda E_i$ $E_i \longleftrightarrow \frac{1}{\lambda} E_i$ $\lambda \ne 0$
  $E_i \longleftrightarrow E_j \pm \lambda E_i$
  $E_i \longleftrightarrow E \pm \frac{1}{\lambda} E_i$

- Condicionamiento de una matriz
  $k(A) = \|A\| \, \|A^{-1}\| \Rightarrow k = 1$ ✓
  $k > 1$ mal, si $>$ peor

- Diagonal $x_i = \frac{b_i}{a_{ii}}$

- $A = \text{Tridiag.} \Rightarrow L$ y U $\Rightarrow$ Tridiagonales

$|\text{Tridiag}| = \prod a_{ii}$

- Resolución progresiva $x_1 = \frac{b_1}{a_{11}}$ $x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j}{a_{ii}}$
  $Lx = b^*$

- Resolución regresiva $x_n = \frac{b_n}{a_{nn}}$ $x_i = \frac{b_i - \sum a_{ij} x_j}{a_{ii}}$
  $Ux = b$

$\Rightarrow c_{ij} = \sum_{k=1}^{\min(i,j)} l_{i,k} u_{k,j}$

GAUSS
$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right)$
① $a_{ii} \ne 0$, si no $E_i \leftrightarrow E_j$
② $m_{n1} = \frac{a_{n1}}{a_{11}}$ $E_n' = E_n - m_{n1} E_1$
③ Repetir hasta U
Sol $x_n^* = \frac{b_n^*}{a_{nn}^*}$ $x_i = \frac{b_i - \sum_{j=i}^{n} a_{ij} x_j}{a_{ii}}$

GAUSS – JORDAN
$|A| \ne 0$
$Ax = b \rightarrow Dx = b^*$ Sol $\Rightarrow x_i = \frac{b_i}{a_{ii}}$
① y ②
③ $m_{12} = \frac{a_{12}}{a_{22}}$ $E_1'' = E_1 - m_{12} E_2'$
ceros hacia arriba
④ Repetir

PIVOTAJE
- PARCIAL
  $\max |a_{ir}|$ $i \ge r$, r, n,
  $E_i \leftrightarrow E_j$
- TOTAL
  $\max |a_{ij}|$, $i \ge r$, $j \ge r$
  – Filas y columnas no tratadas
  – permuta de fila y columna
  ¡¡ Permuta la Sol !!

$A \Rightarrow LU \Rightarrow L \underbrace{Ux}_{y} = b \rightarrow Ly = b \rightarrow (y)$
$\qquad \rightarrow Ux = y \rightarrow (x)$

TRIDIAGONAL
$A = LU / u_{ii} = 1 \Rightarrow$ CROUT
$\begin{pmatrix} \alpha_1 & & \\ \beta_2 & \alpha_2 & \\ & \ddots & \ddots \\ & \beta_{n} & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & \delta_1 & & \\ & 1 & \delta_2 & \\ & & \ddots & \delta_{n-1} \\ & & & 1 \end{pmatrix}$
$A \begin{cases} |a_1| > c_1 > 0 \\ |a_k| \ge |b_k| + |c_k| \\ |a_n| > |b_n| > 0 \end{cases}$ $b_k c_k \ne 0$, $k = 2 \dots n$

DOOLITTLE
$l_{ii} = 1$, $u_{ii} \ne 0$
$|A| \ne 0$
① 1ª Fila de $L \times U$
② (L - 1ª Fila) $\times$ (1ª columna U)
③ 2ª Fila de $L \times U$
④ (L - 1ª y 2ª Fila) $\times$ (2ª columna U)
⑤ (3ª Fila de L) $\times$ (U - 1ª y 2ª COLUM)
⑥ $Ly = b$, $Ux = y$

CROUT
$u_{ii} = 1$; $l_{ii} \ne 0$
$|A| \ne 0$
Cambio la $U \times$ la L y es Doolittle pero con colum $\leftrightarrow$ Fila
(l_{ii})

$A = A^T$
$A = LDL^T = LU / l_{ii} = 1$
Doolittle

CHOLESKY Condiciones $\rightarrow$ • $A = A^T$ • Def ⊕
① 1ª FILA $L \times L^T$
② 2ª FILA $L \times$ (L^T - 1ª COLUM)
③ 3ª FILA $L \times$ (L^T - 1ª y 2ª COLUM)
④ $Ly = b$, $L^T x = y$
⑤ $|A| = \prod_{i=1}^{n} l_{ii}^2$

DOO $\rightarrow$ CROUT
$U \Rightarrow DU' / u'_{ii} = 1 \rightarrow A = LDU' = L'U' / u'_{ii} = 1$, $l'_{ii} \ne 0$

DOO $\rightarrow$ CHOLESKY Cond. de Cholesky
$U = DU' \rightarrow D = D^{1/2} D^{1/2}$
$A = LU = \underbrace{LD^{1/2}}_{L'} \underbrace{D^{1/2} U'}_{L'^T} \Rightarrow A = L'L'^T / l_{ii} > 0$

Si mal cond.
④ Comparar con cotas
⑤ Refinamiento

COTAS
$b \begin{cases} \frac{\|r\|}{k(A)\|b\|} \le & \frac{k(A)\|r\|}{\|b\|} \end{cases}$
$A \begin{cases} \frac{\|R\|}{k(A)\|A\|} \le \frac{\|x_e - x\|}{\|x\|} \le \frac{k(A)\|R\|}{\|A\|} \end{cases}$
b y $A \begin{cases} \frac{\|x_e - x\|}{\|x\|} \le \frac{k(A)\|I\|}{1 - k(A)\frac{\|R\|}{\|A\|}} \left(\frac{\|R\|}{\|A\|} + \frac{\|r\|}{\|b\|} \right) \end{cases}$

$r = Ax_e - x$ $\rightarrow$ Impre b, se está
$x_e = A^{-1}(b + \Delta b)$ en $A \Rightarrow R$
$x = A^{-1} b$

REFINAMIENTO
① Resolver $Ax = b \rightarrow x_0$ ② residuo $|x_0 - x| = e_0$; $r_0 = Ax_0 - b$ ③ Error $Ae_0 = r_0 \Rightarrow e_0$ ④ $x_1 = x_0 + e_0$ ⑤ $\|r_i\| \ge \|r_{i-1}\| \Rightarrow$ Fin; $\|r_i\| < \|r_{i-1}\|$ volver a ③

ITERATIVOS
$Ax = b \rightarrow Ax + x = b + x \rightarrow (A + I)x = b + x \rightarrow B = (A + I)$, $C = -b$ $\Rightarrow x = Bx + C$
$G = (I - B) A^{-1}$ (Cond. consis)
Si $\exists \|B\| < 1$ ó $\rho(B) = \max|\lambda_i(B)| < 1$ $\Rightarrow A = M - N$ $\begin{cases} M \Rightarrow \exists M^{-1} \\ N \Rightarrow \{D, U, L\} \end{cases}$ $x^{n+1} = M^{-1} N x^n + M^{-1} b$
La 'm' utilizada en los métodos es la 'n' de $A^{n \times n}$

JACOBI
$A = D - L - U$
$M = D$
$N = L + U$
$x^{n+1} = D^{-1}(L + U)x^n + D^{-1} b$
$C_j = D^{-1} b$
$B_j = I - D^{-1} A$ ó $B_j = D^{-1}(L + U)$
$x_i^{n+1} = \frac{b_i - \sum_{j=1, j \ne i}^{n} a_{ij} x_j^n}{a_{ii}}$ $i = 1, 2, \dots, n$

GAUSS – SEIDEL
$M = D - L$
$N = U$
$A = D - L - U$
$x^{n+1} = (D - L)^{-1} U x^n + (D - L)^{-1} b$
$x^{n+1} = Bx^n + C$ $\begin{cases} B_{G-S} = (D - L)^{-1} U \\ C_{G-S} = (D - L)^{-1} b \end{cases}$
$x_i^{n+1} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{n+1} - \sum_{j=i+1}^{n} a_{ij} x_j^n}{a_{ii}}$
$\boxed{Dx^{n+1} = Lx^{n+1} + Ux^n + b}$

W – RELAJACIÓN $w = 1 \rightarrow GS$ < 1 SUB > 1 SOBRE
$A = D - L - U \Rightarrow$ $M = \frac{D}{w} - L$, $N = \frac{D}{w} - D + U$
$Dx^{n+1} = w(b + Lx^{n+1} + Ux^n) + (1 - w) Dx^n \Rightarrow$ Exp. matricial
$x_i^{n+1} = \frac{w\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{n+1} - \sum_{j=i+1}^{n} a_{ij} x_j^n\right]}{a_{ii}} + (1 - w) x_i^n$

CONV
(PART) ① Dom diagonal
$\|B_j\| = \max_i \left(\sum_{j \ne i} \frac{|a_{ij}|}{a_{ii}} \right) < 1$
② $A = A^T$ y ⊕ ③ Tridiagonal
SIMPLI
$(\rho(B_j))^2 = \rho(B_{GS})$
$|B_j - \lambda I| = 0 \Rightarrow \{\lambda_i\} \Rightarrow |D^{-1}(L + U) - \lambda I| = 0$
$B_j = D^{-1}(L + U)$
$\exists D \Leftarrow \exists D^{-1} \ne 0$
$|D^{-1}(L + U) - \lambda D^{-1} D| = 0$
$D^{-1} |L + U - \lambda D| = 0$
$|\underbrace{\lambda D - L - U}_{A}| = 0$ signo A ⑤ λ diag
$\rho(B_j) = \max|\lambda_i| < 1 \Rightarrow$ Conver

① = ② = ③ =
SIMPLI
$|B_{GS} - \lambda I| = 0 \Rightarrow \{\lambda_i\}$ $|(D - L)^{-1} U - \lambda I| = 0 \rightarrow |(D - L)^{-1} U - \lambda (D - L)^{-1}(D - L)| = 0$
$B_{GS} = (D - L)^{-1} U$
$\exists (D - L) \Leftarrow \exists (D - L)^{-1} \ne 0$
$|(D - L)^{-1}| \, |U - \lambda D + \lambda L| = 0$
$|U - \lambda D + \lambda L| = 0 \Rightarrow |\lambda D - \lambda L - U| = 0$
$\begin{pmatrix} \lambda a_{11} & a_{12} & a_{13} \\ \lambda a_{21} & \lambda a_{22} & a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{pmatrix}$
$|A| = 0 \Rightarrow \{\lambda_i\}$
$\max|\lambda_i| < 1 \Rightarrow$ conver
$\rho(B_{G-S})$

① Dom diag $0 < w \le 1$ ② $A = A^T$ ⊕ $0 \le w \le 2$
③ = $\Rightarrow$ Valor óptimo $w = \frac{2}{1 + \sqrt{1 - \rho(B_j)^2}}$ $\rho(B_j) = 0 \Rightarrow w = 1$; $\rho(B_j) \ne 0 \Rightarrow$ 1 $< w$
$\boxed{0 \le w \le 2 \text{ necesaria para conv}}$

COTA ERROR $e^n = Be^{n-1}$
$\exists \|B\| < 1 \rightarrow$ ① $x^0 = 0 \rightarrow \frac{\|x^n - x\|}{\|x\|} \le \|B\|^n$
② $x^0 \ne 0 \rightarrow \|x^n - x\| \le \frac{\|B\|}{1 - \|B\|} \|x^n - x^{n-1}\|$
$c = \frac{\|x^n - x^{n-1}\|}{\|x^{n-1} - x^{n-2}\|}$ $\frac{\|x^n - x\|}{\|x\|} \le \frac{\|B\|^n \|x^1 - x^0\|}{1 - \|B\|}$
$\nexists \|B\| < 1$ $\frac{\|x^n - x\|}{\|x\|} \le \frac{C \|x^1 - x^0\|}{1 - C}$

Construcción de polinomio
① Lagrange $P(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$ (forma)
$l_i(x) = \prod_{j=0, j \ne i}^{n} \frac{x - x_j}{x_i - x_j}$ (l_i)
② Newton
$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) \dots$
③ Imponer Cond
$P(x) = a_0 + a_1 x + \dots$
Error $= \frac{f^{n+1}(\xi)}{(n+1)!} \Pi_n(x)$ (Prob interpolación)

f, $\{x_i\}_{i=0}^{n}$ nodos, $[a, b]$ $\boxed{\int_a^b f(x) = \sum_{i=0}^{n} c_i f(x_i) + \int_a^b \frac{f^{n+1}(\xi x)}{n+1!} \Pi_n(x) dx}$ $\rightarrow$ Si $\{x_i\}_{i=0}^{n}$ equidistantes $\Rightarrow$ NEWTON – COTES

- Valor medio integral $\rightarrow$ Sea $f, g \in C([a, b]) / g$ no cambia de signo en $[a, b] \Rightarrow \exists \theta \in [a, b] / \int_a^b f(x) g(x) dx = f(\theta) \int_a^b g(x) dx$

NEWTON – COTES | Nodos equidistantes, h tamaño entre nodos Cerradas $\rightarrow x_0 = a$ y $x_n = b$ Abiertas $\rightarrow x_0 = a + h$ $hx_n = b$

Trapecio $\rightarrow$ $x_0 = a$, $x_1 = b$ $h = b - a$ $\frac{b - a}{2}(f(a) - f(b))$ (Error) $\frac{(b - a)^3}{12} f''(\theta)$ (FORMU) $\int_a^b f = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{h^3}{12} f''(\theta)$

– Tabla

Nodos	a	b
Pesos	$\frac{b-a}{2}$	$\frac{b-a}{2}$

FORMULA SIMPSON $x_0 = a$ $x_1 = \frac{a + b}{2}$ $x_2 = b$ $\boxed{n = 2}$ Utiliza $P_n(x)$ Newton $f[x_0] = f(x_0)$
$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$

$\int_a^b P_2(x) = \frac{f(x_0) + 4f(x_1) + f(x_2)}{3} h$ – Tabla

a	$\frac{a+b}{2}$	b
$\frac{b-a}{6}$	$\frac{2(b-a)}{3}$	$\frac{b-a}{6}$

(Error) $\frac{f^{IV}(\theta)}{90} h^5$ $\theta \in [a, b]$

$f[x_0, x_1, x_2] = \frac{f(x_2) - 2f(x_1) + f(x_0)}{2h^2}$

$\int_a^b f = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)) - \frac{f^{IV}(\theta)}{90} h^5$
$x_0 = a$, $x_2 = b$, $x_1 = x_0 + h = \frac{a + b}{2}$ $h = \frac{b - a}{2}$

- Fin de tipo interpolatorio de $n+1$ nodos (n subintervalos) son $\begin{cases} \text{orden } n \text{ si } n+1 \text{ par} \\ \text{orden } n+1 \text{ si } n+1 \text{ impar} \end{cases}$

COMPUESTAS

$$\int_a^b f = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f = \sum_{i=1}^{N}\sum_{j=0}^{N} c_{ij} f(x_{ij}) + \sum_{i=1}^{N} R_i(f)$$

Teorema valor medio $\exists\theta \,/\, f(\theta) = \frac{f(a)+f(b)}{2}$

GRAM-SCHMIDT

$w_1 = v_1$

$w_2 = v_2 - \frac{\langle w_1, v_2\rangle}{\langle w_1, w_1\rangle} w_1$

$\boxed{\text{Trapecio compuesta}}$ $[a,b]$, $N =$ nº intervalos, $h = \frac{b-a}{N} =$ cte $=$ long. intervalo $x_i = a + ih$

$$\int_a^b f = \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{N-1} f(x_i) + f(b)\right) - \frac{h^2}{12}(b-a)\underbrace{f^{(ii)}(\theta)}_{\text{Orden 1}}, \quad \theta = [a,b] \qquad \frac{(b-a)^3}{12N^2} f^{(ii)}(\theta) \xrightarrow[N\to\infty]{} 0, \; O(1/N^2) \text{ Precisión 2}$$

$\boxed{\text{Trapecio corregida compuesta}}$

$$\int_a^b f = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right] + \frac{h^2}{12}\left[f'(a) - f'(b)\right] + \frac{h^4(b-a)}{720} f^{(iv)}(\theta) \; ; \; \frac{1}{N}\frac{(b-a)^5}{720} f^{(iv)}(\theta) \quad O\left(1/N^4\right) \quad \theta \in [a,b]$$

$\boxed{\text{simpson compuesta}}$ $[a,b]$, $N =$ nº intervalos, $h = \frac{b-a}{N}$, $x_i = a + ih$

$$\int_a^b f = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f = \sum_{i=1}^{N} \frac{h}{6}\left(f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)\right) - \frac{h^4}{90\cdot 2^5}(b-a) f^{(iv)}(\theta) \; ; \; \frac{(b-a)^5}{90\cdot 2^5}\frac{1}{N^4} f^{(iv)}(\theta) \xrightarrow[N\to\infty]{} 0 \; ; \; O(1/N^4)$$

$\boxed{\text{GAUSS}}$

$n+1$ nodos $\{x_i\}_{i=0}^{n}$

$n+1$ coef. $\{c_i\}_{i=0}^{n}$

No equidistantes.

$$\int_a^b f \simeq \sum_{i=0}^{N} c_i f(x_i) \qquad w(x) = 1$$

$$\int_a^b w f(x) dx = \sum_{i=0}^{n} c_i f(x_i) + \int_a^b w(x) E(f) dx$$

$$\int_a^b w(x) x^k(x) dx \simeq \sum_{i=0}^{n} c_i x^k(x_i) \to \text{Orden } 2n+1$$

$\sum_{i=0}^{n} c_i x_i^k$ $\quad K = 0, 1, \dots, n, n+1, \dots 2n+1$

① Paso $\{x_i\}_{i=0}^{n}$, $\int_a^b w(x) E(f) dx = \int_a^b w(x) \frac{f^{(n+1)}(\xi_x)}{n+1!} \underbrace{(x-x_0)(x-x_1)\dots(x-x_n)}_{\pi_n(x) \text{ grado } n+1} dx$

$f = P + E(f) = \frac{f^{(n+1)}(\xi_x)}{n+1}\pi_n(x) \to \int_a^b w f = \sum c_i f(x_i) + \int_a^b w E(f)$

$$R(x^k) = \int_a^b \frac{w(x)}{(n+1)!}(x^k)^{(n+1)} \pi(x) dx = 0$$

② Calculo $B' \perp B = \{1, x, x^2, \dots, x^{n+1}\}$ con $(f,g) = \int_a^b f(x) g(x) w(x) dx$ de B' tomo w_{n+1} y $P_{n+1} = w_{n+1}$ y tomo $P_{n+1} = 0$ y calculo sus raíces $\mathbb{R}$ distintas y están en $[a,b]$. Los nodos $n+1$ de Gauss $\{x_0, \dots, x_n\}$

③ Error $\to$ Polinomio de Hermite P_{2n+1} $\exists$ $f = P_{2n+1} + E(f)$

$$\int_a^b w(x) f(x) = \sum_{i=0}^{n} c_i f(x_i) + \int_a^b w(x) \underbrace{\frac{f^{(2n+2)}}{(2n+2)!}}_{\exists\theta\in[a,b]} \underbrace{(x-x_0)(x-x_0)\dots(x-x_n)(x-x_n)}_{\pi_n^2(x) \geq 0}$$

NODOS: $x_0, x_1, \dots, x_n$

COND. $\boxed{\begin{matrix} f(x_0), f(x_1), \dots f(x_n) \\ f'(x_0), f'(x_1), \dots f'(x_n) \end{matrix}}$ $2n+2$

$$E(f) = \frac{f^{(2n+2)}(\theta)}{(2n+2)!} \int_a^b w(x) \pi_n^2(x) dx$$

Sol. final $\to$ $\int_a^b w(x) f(x) dx = \sum_{i=0}^{n} c_i f(x_i) + \frac{f^{(2n+2)}(\theta)}{(2n+2)!}\int_a^b w(x) \pi_n^2(x) dx$ $\quad$ $n+1$ NODOS $[a,b]$, $w(x) \geq 0$ $\quad$ $\boxed{\text{FORMULA GAUSS}}$

$\boxed{\text{continuo}}$ Def: $f: [a,b] \times \mathbb{R} \to \mathbb{R}$ Lipchitziana respecto a la segunda variable: ⓐ Si $\frac{\partial f}{\partial y}(x,y)$ $\exists$, continua y acotada

$\exists!$ sol $\Rightarrow$ $f: [a,b] \times \mathbb{R} \to \mathbb{R}$ continua y Lipchitziana respecto a la 2ª variable $\Rightarrow$ (P) $\exists!$ Sol en $[a,b]$ $y(x) \in C^1([a,b] \subset \mathbb{R})$

① Prob de valor inicial $\to$ PVI $\to$ EDO con valor inicial (P) $\begin{cases} y'(x) = f(x, y(x)) & \forall x \in [a,b] \\ y(x_0) = y_0 \end{cases}$ $\boxed{1}$

$\boxed{\begin{matrix} L > 0 \to |f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2| \\ \forall x \in [a,b] \quad \forall y_1, y_2 \in \mathbb{R} \end{matrix}}$

② Sistemas lineales de EDO

$$\left.\begin{matrix} y_1'(x) = f_1(x, y_1, y_2, \dots, y_n), \; y_1(x_0) = y_{10} \\ \vdots \\ y_m(x) = f_m(x, y_1, y_2, \dots y_n), \; y_m(x_0) = y_{m0} \end{matrix}\right\} \to \begin{matrix} \bar{y}'(x) = \bar{f}(x, \bar{y}) \\ \bar{y}(x_0) = \bar{y}(x_0) \end{matrix}$$

EULER ESCALAR

$(E) = \begin{cases} y_{n+1} = y_n + h f(x_n, y_n) \\ y_0 \end{cases}$

③ Orden de la EDO $\to$ Un PVI de orden m $\Rightarrow$ sistema lineal de dim. m, de ecuaciones diferenciales ordinarias $\Rightarrow$ $\Rightarrow$ PVI de orden 1 en forma vectorial $\Rightarrow y^{m}(x) = f(x, y(x), y'(x), \dots, y^{m-1}(x))$ $x \in [a,b]$, $y(x_0), y'(x_0), \dots, y^{m-1}(x_0)$

- $z_1(x) = y(x)$, $z_2(x) = y'(x)$, $\dots$, $z_{m-1}(x) = y^{m-2}(x)$, $z_m(x) = y^{m-1}(x)$;

$z_m'(x) = f(x, z_1(x), \dots, z_m(x))$ $\quad$; $z_m(x_0) = y_{m0}$

$z_{m-1}'(x) = z_m(x)$ $\quad$; $z_{m-1}(x_0) = y_{m-1\,0}$

$\vdots$

$z_2'(x) = z_3(x)$ $\quad$; $z_2(x_0) = y_{20}$

$z_1'(x) = z_2(x)$ $\quad$; $z_1(x_0) = y_{10}$

Sistema lineal EDO con ci en x_0

$\boxed{\text{DISCRETO}}$ Métodos de discretización $\to$ Dominio $[a,b]$ $\to$ malla de tamaño de paso h o bien $N = \frac{b-a}{h} =$ nº nodos/pasos

$x_n = x_0 + nh$; $b - a = N \cdot h$ luego $h \to 0 \equiv N \to \infty$ $\quad$ (P) $\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$ $\xrightarrow{\text{discretizo}}$ $(P_h) = \begin{cases} y_{n+k} = \mathcal{F}(x_n, \dots, x_{n+k}, y_n, \dots, y_{n+k}) \\ y_0, y_1, \dots, y_k \text{ [Necesito } k \text{ valores de arranque]} \end{cases}$

$y(x)$ sol exacta $\quad$ $\{y_n\}_{n=0}^{N}$ Sol aprox $\quad$ $y_n \simeq y_n$, $n = 0, \dots, N$

ⓐ Métodos de 1 paso ($K=1$)

$$\left.\begin{cases} y_{n+1} = \mathcal{F}(x_n, x_{n+1}, y_n, y_{n+1}) \\ y_0 \end{cases}\right\} \to \text{Sol} \begin{cases} \boxed{\text{Explícitos}} \\ \text{Cada iter} \to 1 \text{ evaluación} \\ 1 \text{ valor de arranque} \\ \boxed{\text{Implícitos}} \to y_{n+1} = \mathcal{F}(\underbrace{x_n, x_{n+1}, y_n}_{\text{Conocidos}}, y_{n+1}) = \mathcal{F}'(y_{n+1}) \end{cases}$$

$\boxed{\text{Punto fijo}}$ $\Rightarrow$ $\mathcal{F}$ CONTRACTIVA $\begin{cases} y_{n+1}^{p+1} = \mathcal{F}'(y_{n+1}^{p}) \\ y_{n+1} = y_n \end{cases}$ $\quad$ $\{y_{n+1}^{p}\} \xrightarrow[p\to\infty]{} y_{n+1}$

ⓑ K Pasos $\Rightarrow$ $y_{n+k} = \mathcal{F}(x_n, \dots, x_{n+k}, y_n, \dots, y_{n+k})$

$\boxed{\text{Explícitos}}$

- 1 evaluación de $\mathcal{F}$ / 1 iteración
- K valores de arranque

$\boxed{\text{IMPLÍCITOS}}$

- 1 método iterativo / 1 iteración
- K valores de arranque

① $\to$ (P_h) orden p $\Leftrightarrow$ $\frac{e_n}{h} = O(h^p)$

② $\to$ (P_h) consistente $\Leftrightarrow$ $\frac{e_n}{h} = O(h) \Leftrightarrow O(1) \Leftrightarrow \phi(x,y,0) = f(x,y)$

③ $\to$ Si (P_h) estable y de orden p es ①, (P_h) converge y $err \xrightarrow[h\to 0]{} 0$ a la velocidad $p \equiv \max_{0\leq n\leq N} |err| \leq K \cdot h^p \equiv$ $\equiv \max_{0\leq n\leq N} |err| = O(h^p)$

Teorema $\to$ Sea $f \in C^p([a,b] \in \mathbb{R})$, $\phi \in C^p([a,b] \times \mathbb{R} \times [0, h_0]) \Rightarrow$

$$P_h \text{ orden } p \Leftrightarrow \begin{cases} \phi(x,y,0) = f(x,y) \\ \frac{\partial \phi}{\partial h}(x,y,0) = \frac{1}{2} f'(x,y) \\ \vdots \\ \frac{\partial^{p-1}\phi}{\partial h^{p-1}}(x,y,0) = \frac{1}{p} f^{(p-1)}(x,y) \\ \frac{\partial^{p-1}\phi}{\partial h^{p}}(x,y,0) = \frac{1}{p+1} f^{p}(x,y) \end{cases}$$

Métodos de un paso

ⓐ y ⓑ → Euler y métodos de resolución de PVI

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n,y_n) + f(x_{n+1}, \underbrace{y_n + h f(x_n,y_n)}_{\simeq y_{n+1}})\right]$$

Sea $\begin{cases} y'(x) = f(x,y(x)) \\ + \\ \text{fin} \end{cases}$ → * Malla (x_n)

* $\int_{x_n}^{x_{n+1}} y'(x)dx = \int_{x_n}^{x_{n+1}} f(x,y(x)) \simeq \text{fin}[f,[x_n,x_{n+1}]]$

$y(x_{n+1}) \simeq y_{n+1} \Rightarrow y(x_n) \simeq y_n$

* Imponer igualdad $y_{n+1} = y_n + \text{fin}[f,[x_n,x_{n+1}]]$

ⓒ Estudio general

* (P) $\begin{cases} y'(x) = f(x,y(x)) & x \in [a,b] \\ y(x_0) = y_0 \end{cases}$ * Malla $x_n = a + nh,\ n = 0,\dots,N$

* $(P_n) = \begin{cases} y_{n+1} = y_n + h\,\phi(x_n,y_n,h), & n = 0 \dots N \\ y_0 \end{cases}$

$\phi(x,y,h)$ = Función que define el método

* Estudio Error $|y_n - y(x_n)| \to n = 0,\dots,N \to \max|y_n - y(x_n)|$ $0 \le n \le N$
* Bondad del método $\begin{cases} \text{- Consistencia: Error en la iteración } n+1 \\ \text{- Estabilidad: Influencia/propagación del error} \end{cases}$ → Convergencia: $\max\limits_{0\le n\le N} |y_n - y(x_n)| \xrightarrow[N\to\infty]{} 0$
* Velocidad de Convergencia → orden $(1/N \to 0,\ 1/N^2 \to 0)$ ① ②

ERROR → GLOBAL ⇒ $err = y(x_n) - y_n$
→ LOCAL ⇒ Error en una iteración → $R_{n+1} = y(x_{n+1}) - [y(x_n) + h\,\phi(x_n, y(x_n), h)]$

→ $\{y_n\}_{n=0}^{N}$ → Verifica (P_n) " $y_{n+1} = [y_n + h\,\phi(x_n,y_n,h)] =$
→ $\{y(x_n)\}_{n=0}^{N}$ → Verifica (P) con error R_{n+1}
→ $y(x_{n+1}) - [y(x_n) + h\,\phi(x_n, y(x_n), h)] = R_{n+1}$

CONVERGENCIA → $\max|y(x_n) - y_n| \xrightarrow[h\to 0]{} 0$ con h y N fijo

CONSISTENCIA → Con ① si: $\sum_{n=0}^{N} |R_n| \xrightarrow[h\to 0]{} 0 = \max\limits_{0\le n\le N} \frac{|R_n|}{h} \xrightarrow[h\to 0]{} 0 \equiv \max\limits_{0\le n\le N}\left|\frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n), h)\right| \xrightarrow[h\to 0]{} 0$

ESTABILIDAD

Sea $\{y_n\}$ Sol de $\begin{cases} y_{n+1} = y_n + h\,\phi(x_n,y_n,h) \\ y_0 \end{cases}$ $\{z_n\}$ Sol de $\begin{cases} z_{n+1} = z_n + h\,\phi(x_n,y_n,h) + err \\ z_0 \end{cases}$

(P_n) estable ⇒ ∃ M cte independ. de h

$\max\limits_{0\le n\le N}|y_n - z_n| \le M\left[|y_0 - z_0| + \sum_{n=0}^{N} |err|\right]$

* Si (P_n) es estable y consistente ⇒ Converge

ORDEN → (P_n) es de orden p si p es el mayor natural que: $\underbrace{\max\limits_{0\le n\le N} \left|\frac{R_n}{h}\right|}_{} \le k h^p \quad \forall y$ sol ①

↳ $\max\limits_{0\le n\le N}\left|\frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n), h)\right|$

Con k cte depende de ϕ, no de h

Def: $\boxed{f(h) \in O(h^p) \text{ si } \frac{f(h)}{h^p} \xrightarrow[h\to 0]{} 0}$

• Al hueco de la anterior

UN PASO ORDEN SUPERIOR

ⓓ Objetivo → método m-n-1 paso explícito de orden superior

*** MÉTODO DE TAYLOR** → ① $y' = f(x,y(x)),\ f \in C^{p-1}$; $y(x+h) = y(x) + h\underbrace{y'(x)}_{f(x,y)} + \frac{h^2}{2}\underbrace{y''(x)}_{f'(x,y)} + \dots + \frac{h^p}{p!}\underbrace{y^{p)}(x)}_{f^{p-1)}(x,y(x))} + O(h^{p+1}) =$

$= y(x) + h\underbrace{\left[f(x,y(x)) + \frac{h}{2} f'(x,y(x)) + \dots + \frac{h^{p-1}}{p!} f^{p-1)}(x,y(x))\right]}_{\phi(x,y,h)} + O(h^{p+1})$

$(P_n)\begin{cases} y_{n+1} = y_n + h\,\phi(x,y_n,h) \\ y_0 \end{cases}$ MÉTODO TAYLOR

CONSISTENCIA Y ORDEN: $x = x_n$, $x + h = x_{n+1}$ → $\left|\frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n,y_n,h)\right| = \frac{O(h^{p+1})}{h} = O(h^p)$ ⇒ (P_n) orden p Error local

ESTABILIDAD → (P_n) estable ⟺ $\phi(x,y,h)$ Lipschitziana 2ª variable ‖ Para cada f, derivar $(f', f'', \dots, f^{k)})$ Comprobar Lipschitziana 2ª variable

RUNGE-KUTTA R EVALUACIONES → Comparar estrategias numéricas: Obtener f.i.n./método de resolución PVI

$$\phi(x,y,h) = \sum_{r=1}^{R} \underbrace{C_r}_{\text{Pesos}}\ \underbrace{K_r(x,y,h)}_{\text{Evaluaciones}}$$

$\begin{cases} K_1 = f(x,y) \\ K_2 = f(x + h a_2, y + h b_{21} f(x,y)) \\ \vdots \\ K_r = f(x + h\underbrace{a_r}_{\text{parámetros}}, y + h \sum_{s=1}^{r-1} \underbrace{b_{rs} k_s}_{\text{parámetros}}) \end{cases}$ $r = 1,\dots,R$

a_r, b_{rs}, C_r → elegidos para conv. y orden max

CONSISTENCIA Sea f Lipsch... a 2ª variable

$K_r(x,y,0) = f(x,y),\ \forall r \Rightarrow \phi(x,y,0) = \sum_{r=1}^{R} C_r \underbrace{K_r(x,y,0)}_{f(x,y)} =$

$= f(x,y)\sum_{r=1}^{R} C_r \stackrel{?}{=} f(x,y)$ $\boxed{\sum_{r=1}^{R} C_r = 1 \Rightarrow \text{Consistencia}}$

ESTABILIDAD → f Lipschitziana 2ª var ⇒ $K_1 = f$ Lipschit 2ª var ¿K_2 Lipch?

$|k_2(x,y_1,h) - k_2(x,y_2,h)| \underset{\exists L}{=} |f(x + ha, y_1 + hb f(x,y_1)) - f(x + ha, y_2 + hb f(x,y_2))| \le L|y_1 + hb f(x,y_1) - (y_2 + hb f(x,y_2))| \le$

$\le L(|y_1 - y_2| + hb|f(x,y_1) - f(x,y_2)|) \le \underbrace{L(1 + hbL)}_{M = cte}|y_1 - y_2| \Rightarrow k_2$ Lipsch

Si f Lipch de 2ª vari $= \begin{cases} \text{ESTABLE} \\ \text{CONVERGE} \end{cases} \Longleftrightarrow \sum_{r=1}^{R} C_r = 1$

ORDEN → P(R) = Orden max de Runge-Kutta de R evaluaciones

• Si $R = 1,2,3,4 \to P(R) = R$ • Si $R = 5,6,7 \to P(R) = R-1$ • Si $R \ge 8 \to P(R) = R-2$

$\boxed{\frac{|e_n|}{\text{err global}} \le K h^4 \quad R\text{-}K4}$

R-K2 orden 2 converge $\begin{cases} C_1 + C_2 = 1 \\ C_2 \cdot a_2 = 1/2 \\ C_2 \cdot b_{21} = 1/2 \end{cases} \Rightarrow C_2[a_2 f_x + b_{21} f f_y]$

EJERCICIOS Y EJEMPLOS ⓐ $f'(x,y) = f_x + f f_y$ $\quad \frac{\partial\phi}{\partial h} = \frac{1}{2} f'(x,y) \Rightarrow$ orden 2 $\quad$ si $\phi(x,y,0) = f(x,y) \Rightarrow$ orden 1

① $\begin{cases} y'(x) = y(x) - y^2(x) + \int_0^x y(s)ds \\ y(0) = 1 \end{cases}$ ⓑ $\Rightarrow \partial$ ① $\Rightarrow \begin{cases} y''(x) = f(x, y(x), y'(x)) \\ y(x_0) = y_0 \\ y'(x_0) = y'_0 \end{cases}$

$\Rightarrow y(0) = 1$

sustit. → $y'(0) = y(0) - y^2(0) + \int_0^0 y(s)ds = 1 - 1 + 0 = 0$

derivada → $y''(t) = y'(t) - 2y(t)y'(t) + y(t)$ $\begin{cases} y_1(t) = y(t) \\ y_2(t) = y'(t) \end{cases} \Rightarrow y_2' = y''(t)$

$y_1'(t) = y_2(t)$

$y_1(0) = 1$

$y_2(0) = 0$

$y_2'(t) = y_2(t) - 2y_2(t)y_1(t) + y_1(t)$

$\bar{y}(t) = (y_1(t), y_2(t))$

$\bar{F}(t) = (f_1(t), f_2(t))$

$\Rightarrow \begin{cases} \bar{y}'(t) = (y_2(t),\ y_2(t) - 2y_1(t)y_2(t) + y_1(t)) \\ \bar{y}'(0) = (1,0) \end{cases}$

ⓒ

m $\begin{cases} \bar{y}_{n+1} = \bar{y}_n + h f(x_n + \alpha h, \bar{y}_n + \alpha h f(x_n, \bar{y}_n)) = \bar{y}_n + \bar{F}(x_{n+1/2}, \bar{y}_n + 1/2\, \bar{F}(x_n, \bar{y}_n)) \\ \bar{y}_0 = (y_1^0, y_2^0) \end{cases}$; $\bar{y}_n = (y_1^n, y_2^n)$

$\begin{cases} \bar{y}_{n+1} = \bar{y}_n + \bar{F}(x_{n+1/2}, \bar{y}_n + 1/2(y_2^n, y_2^n - 2y_1^n y_2^n + y_1^n)) \\ \bar{y}_0 = (1,0) \end{cases}$

$$\underbrace{\bar{y}_n + 1/2\left(y_2^n, y_2^n - 2y_1^n y_2^n + y_1^n\right)}_{} \Rightarrow \left(\underbrace{y_1^n + 1/2\, y_2^n}_{z_1}, \underbrace{y_2^n + 1/2\, y_2^n - 2y_1^n y_2^n + y_1^n}_{z_2}\right)$$

$$\begin{cases} y_1^{n+1} = y_1^n + z_2^n \\ y_2^{n+1} = y_2^n + z_2^n - 2 z_1^n z_2^n + z_1^n \\ y_1^0 = 1 \\ y_2^0 = 0 \end{cases} \Rightarrow \begin{matrix} y_1^1 = 3/2 \\ y_2^1 = 1/2 \end{matrix}$$

Sol $\simeq y(1) = y_1(1) \simeq y_1^1 = 3/2$

$$(P)\begin{cases} y' = x - y & x \in [0,1],\ y \in \mathbb{R} \\ y(0) = 1 \end{cases}$$

ⓐ $y' = x - y \to y' + y = x \to$ EDO lineal de coefici. constantes no homoge

① Sol homogenea asociada $\to y' + y = 0 \to \lambda + 1 = 0 \to \lambda = -1 \Rightarrow e^{-x} c$ Sol

② Sol general de la EDO $\to y(x) = Ax + B \to A + Ax + B = x \Rightarrow$ Sol particular $\Rightarrow y(x) = x - 1$

$$\begin{cases} y(x) = c e^{-x} + x - 1 \\ y(0) = 1 \Rightarrow c = 2 \\ \boxed{y(x) = 2e^{-x} + x - 1} \end{cases}$$

ⓑ Euler

$$(P)\begin{cases} y' = \overbrace{x - y}^{f(x,y)},\ x \in [0,1] \\ y(0) = 1 \end{cases} \quad (E) = \begin{cases} y_{n+1} = y_n + h f(x_n, y_n) \\ y_0 = 1 \end{cases}$$

$h = 0'2 \Rightarrow N = \frac{b-a}{h} = 5 \Rightarrow$ Malla $x_0 = 0, x_1 = 0'2 \ldots x_5 = 1$

$x_n = a + hn \qquad x_n = n\,0'2,\ n = 0, \ldots, 5$

$y_{n+1} = y_n + 0'2 f(0'2n, y_n) = y_n + 0'2(0'2n - y_n) = (0'2)^2 n + 0'8 y_n$

Heum $y_{n+1} = y_n + h\Big[\underbrace{1/2}_{c_1} \underbrace{f(x_n, y_n)}_{k_1} + \underbrace{1/2}_{c_2} \underbrace{f(x_n + h, y_n + h f(x_n, y_n))}_{k_2}\Big] \Rightarrow h = 0'2$

$\emptyset = c_1 k_1 + c_2 k_2 \,/\, c_1 + c_2 = 1 \Rightarrow$ orden 1

$$(P_n)\begin{cases} y_{n+1} = y_n + 0'2\, 1/2\left((x_n - y_n) + x_n + 0'2 - (0'8 y_n + (0'2)^2 n)\right) = y_n + 0'1\left[2(0'2n) - (0'2)^2 n + 0'2 - 1'8 y_n\right] = \\ \qquad = y_n + 0'1\left[0'36n + 0'2 - 1'8 y_n\right] \\ y_0 = 1 \end{cases}$$

ⓐ
$$\begin{cases} G(x) = (J - A)x + b \Rightarrow \text{Contrac} \\ G(y) = (J - A)y + b \end{cases}$$

$$\|G(x) - G(y)\| = \|(J - A)x + \cancel{b} - (J - A)y - \cancel{b}\| = \|\underline{(J - A)(x - y)}\| \le \underbrace{\|(J - A)\|}_{<1} \|(x - y)\|$$

ⓑ Tiene Sol y es única? $x = A^{-1} b$? $\Rightarrow x^n = G(x^{n-1}) = (J - A)x^{n-1} + b$

$x = G(x) \Rightarrow x = (J - A)x + b;\ x - x = -Ax + b;\ Ax = b$

$A = M - N \to A = D - L - U \quad \begin{cases} M = D \\ N = L + U \end{cases} \qquad M^{-1}N = D^{-1}(L + U) = D^{-1}(D - A) = \underline{J - D^{-1}A}$

$L + U = D - A$

$B_w = D^{-1}(D - wA)$

$C_w = D^{-1} w A)$

ⓑ $\displaystyle\int_{-1}^{1} |x| f(x)\,dx \simeq A_0 f(x_0) + A_1 f(x_1)$

x_0 y $x_1 \to$ raices de $\to a + bx + cx^2 = \pi(x)$ $\begin{cases} \int_{-1}^{1} \pi(x)\,dx = 0 \\ \int_{-1}^{1} x\,\pi(x)\,dx = 0 \end{cases} \Rightarrow \pi(x) = x^2 - 1/2$

$B \to \{1, x\}$ $\quad f(x) = 1 \to \int_{-1}^{1} |x|\,dx = 1 = A_0 + A_1$; $\quad f(x) = x \to \int_{-1}^{1} |x| x\,dx = 0 \to A_0(-1/\sqrt{2}) + A_1(1/\sqrt{2})$

$\displaystyle\int_{-1}^{1} |x| f(x)\,dx \simeq 1/2\{f(-1/\sqrt{2}) + f(1/\sqrt{2})\}$ ⓑ $\displaystyle\log 2 = \int_0^1 \frac{2x}{1+x^2}\,dx$ $\quad \displaystyle\int_{-1}^{1} \frac{|x|}{1+x^2}\,dx = 2\int_0^1 \frac{|x|}{1+x^2} = \int_0^1 \frac{2x}{1+x^2}\,dx = \log 2$

$f(x) = \frac{1}{1+x^2} \to$ aplico fórmula

GAUSS 3 Puntos $\{x_0, x_1, x_2\}$ $n = 2 \Rightarrow \displaystyle\int_a^b f = \sum_{i=0}^{2} c_i f(x_i) + \int_a^b E_2(f)\,dx \to$ Exacta para $\{1, x, x^2, x^3, x^4, x^5\}$ $\quad (2n+1)$

$\displaystyle L_b \frac{f'''(\xi_x)}{3!} (x - x_0)(x - x_1)(x - x_2)$

① Calcular $\{x_0, x_1, x_2\} \to e(P_k(x)) = 0$; $P_k(x) = x^k$; $k = 3, 4, 5$

$\displaystyle 1/3! \int_a^b (P_k(x))^3 \underbrace{(x - x_0)(x - x_1)(x - x_2)}_{q(x)}\,dx \to x_0, x_1, x_2$ raices de $q(x)$

② Calcular $\{c_i\}_{i=0}^{2}$ $\quad \displaystyle\int_a^b x^k = \sum_{i=0}^{2} c_i x_i^k$, $k = 0, 1, 2$

$$\begin{cases} k = 0 \to b - a = c_0 + c_1 + c_2 \\ k = 1 \to \frac{b^2 - a^2}{2} = c_0 x_0 + c_1 x_1 + c_2 x_2 \\ k = 2 \to \frac{b^3 - a^3}{3} = c_0 x_0^2 + c_1 x_1^2 + c_2 x_2^2 \end{cases} \Rightarrow \text{Sol } \int_a^b f \simeq c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$$